10H; QNo 23H.12 → ~~To state~~ Prove that in a group the identity element is unique.

Or

The identity element in a group is unique.

Ans → Let G be a group and let e be an identity element.

We have to prove that e is unique.

If not, suppose e' be another identity element in a group G.

Since e is the identity element of G, therefore

$$ae = ea = a \quad \text{--- (1)}$$

Similarly, since e' is the identity element of G, therefore,

$$ae' = e'a = a \quad \text{--- (2)}$$

for every $a \in G$.

Since the eqn ① is true for every $a \in G$, and since $e' \in G$, therefore putting $a = e'$ in ①, we have

$$e'e = ee' = e' \quad \text{—— ③}$$

Similarly putting $a = e$ in ②, we have

$$ee' = e'e = e \quad \text{—— ④}$$

Hence from ③ & ④, it follows that,

$$e = e'$$

which means that the identity in a group is unique.

"or"

Proof:– Suppose e and e' are two identity elements of a group G; we have

$ee' = e$; If e' is identity

$ee' = e'$ If e is identity

But ee' is a unique element of G.

Therefore, $ee' = e$ and $ee' = e'$

$\Rightarrow e = e'$

Hence, the identity element is unique.

—o—

QN.→ To prove that,

$(ab)^{-1} = b^{-1}a^{-1}$, where $a, b \in G$.

Ans.→ Let $a, b \in G$ and let their inverses be a^{-1} & b^{-1} respectively, then, $a^{-1}a = e = aa^{-1}$ & $b^{-1}b = e = bb^{-1}$ where e is the identity element

Now, $(b^{-1}a^{-1})(ab) = b^{-1}\{a^{-1}(ab)\}$

$= b^{-1}\{(a^{-1}a)b\}$

$= b^{-1}(eb)$
$= b^{-1}b$
$= e.$

Similarly, $(ab)(b^{-1}a^{-1}) = a\{b(b^{-1}a^{-1})\}$
$= a\{(bb^{-1})a^{-1}\}$
$= a(ea^{-1})$
$= aa^{-1}$
$= e.$

Hence, $b^{-1}a^{-1}$ is the inverse of ab.

9(i) QNo→ To Prove that,
$(ab)^{-1} = b^{-1}a^{-1}$, where $a, b \in G$.
'or'

QNo→ To Prove that $(ab)^{-1} = b^{-1}a^{-1} \forall a, b \in G$ i.e; the inverse of the product of two elements of a group G is the product of the inverses taken in the reverse order.

Ans.→ Let $a, b \in G$ and let their inverses be a^{-1} & b^{-1} respectively, then,

$a^{-1}a = e = aa^{-1}$
& $b^{-1}b = e = bb^{-1}$
where e is the identity element.

Now, $(a,b)(b^{-1}a^{-1}) = [(ab)b^{-1}]a^{-1}$ [$\because$ Composition is associative]
$= [a(bb^{-1})]a^{-1}$ [by associativity]
$= (ae)a^{-1}$ [$\because bb^{-1} = e$]
$= aa^{-1}$ [$\because ae = a$]
$= e$ [$\because aa^{-1} = e$]

similarly again, $(b^{-1}a^{-1})(ab) = b^{-1}[a^{-1}(ab)]$ [by associativity]
$= b^{-1}[(a^{-1}a)b]$
$= b^{-1}(eb)$.

$= b^{-1}b$

$= e$

Thus we have $(b^{-1}a^{-1})(ab) = e = (ab)(b^{-1}a^{-1})$.

$\therefore$ by definition of inverse, we have

$$(ab)^{-1} = b^{-1}a^{-1}.$$

If the group is commutative, then we shall have

$$(ab)^{-1} = a^{-1}b^{-1}.$$

since $b^{-1}a^{-1} = a^{-1}b^{-1}$

—o—

General Properties of a group:-

Q No → If $a, b, c \in G$, then

(i) $ab = ac \Rightarrow b = c$ (left cancellation law)

(ii) $ba = ca \Rightarrow b = c$ (right cancellation law)

"or"

Q No (Cancellation Law)

Let, a, b, c be arbitrary elements of a group G, Then the following are true:

(i) If $ab = ac$ then, $b = c$

(ii) If $ba = ca$ then, $a = b$.

Ans → From, By, Question

$$ab = ac \quad \text{—} \; (1)$$

Let a^{-1} be the inverse of a in G. Then, $aa^{-1} = a^{-1}a = e$

multiplying both sides of (1) by a^{-1} on the left, we have

$$a^{-1}(ab) = a^{-1}(ac).$$

which by associative law becomes

$$(a^{-1}a)b = (a^{-1}a)c$$

or, $eb = ec$ (by definition of Inverse a^{-1})

$\therefore b = c$ (by definition of Identity e)

(ii) From Question

$$ba = ca \quad \text{———} \ (2)$$

Let a^{-1} be the inverse of a in G.
Multiplying both sides of (2) by a^{-1} on the right, we have

$$(ba)a^{-1} = (ca)a^{-1}$$

which by associative law becomes

or, $b(aa^{-1}) = c(aa^{-1})$

or, $be = ce$ (by definition of Inverse a^{-1})

$\therefore b = c$ (by definition of Identity e)

The first & second part of the theorem is proved.

~~a1H Q No → If a, b are elements of a group G, prove that $(ab)^{-1} = b^{-1}a^{-1}$.~~

(or)

Q No → If a is an element of a group (G) then, $(a^{-1})^{-1} = a$.

Ans → Proof :- If e is the identity element, we have

$a^{-1}a = e$ [by definition of Inverse]

$\Rightarrow (a^{-1})^{-1}[a^{-1}a] = (a^{-1})^{-1}e$ [multiplying both on the left by $(a^{-1})^{-1}$ which is necessarily an element of G because a^{-1} is an element of G].

$\Rightarrow [(a^{-1})^{-1} a^{-1}] a = (a^{-1})^{-1}$ [$\because$ Composition in G is associative and e is identity element]

$\Rightarrow ea = (a^{-1})^{-1}$ [$\because (a^{-1})^{-1}$ is inverse of a^{-1}]

$\Rightarrow a = (a^{-1})^{-1}$

$\Rightarrow (a^{-1})^{-1} = a.$

—o—

QNo→ Let a, b, c be three arbitrary elements of a group G:

The following are then true.

(i) If $ac = a$ or $ca = a$ for some element a in G, then c must be the identity element e of G.

(ii) If $ab = e$ or $ba = e$ then a and b are inverses of each other.

Ans.→ (i) ~~Given that~~

~~$ac = a$ — ①~~

Let a^{-1} be the inverse of a in G.

then, $a^{-1}a = aa^{-1} = e$.

If $ac = a$

then, $a^{-1}(ac) = a^{-1}a$ [multiplying on the left both sides by a^{-1}]

$(a^{-1}a)c = a^{-1}a$ [by associative law].

$ec = e$ [by defn of Inverse a^{-1}]

$\therefore c = e$ [by definition of identity e]

Similarly, Let a^{-1} be the inverse of a in G.

If $ca = a$

then, $(ca)a^{-1} = aa^{-1}$ [multiplying on the Right both sides by a^{-1}]

$c(aa^{-1}) = aa^{-1}$ [by associative law]

$ce = e$ [by definition of Inverse a^{-1}]

$\therefore c = e$ [by defn of identity e]

(ii) Let a^{-1} be the inverse of a in G

then, $a^{-1}a = e$

, from Question

$ab = e$

then, $a^{-1}(ab) = a^{-1}e$ [multiplying on the left both sides by a^{-1}]

$(a^{-1}a)b = a^{-1}e$ [by associative law]

$eb = a^{-1}e$ [by defn. of the Inverse a^{-1}]

$\therefore b = a^{-1}$ [by defn of the identity e]

Similarly, Let b^{-1} be the inverse of a in G

then, $bb^{-1} = e$.

$gb = ba = e$

then, $(ba)b^{-1} = eb^{-1}$ [multiplying on the Right both sides by b^{-1}]

$a(bb^{-1}) = eb^{-1}$ [by associative law]

$ae = eb^{-1}$ [by defn of the Inverse a^{-1}]

$\therefore a = b^{-1}$ [by defn of the identity e]

-o-